

CENTRAL LIMIT THEOREMS IN RANDOM CLUSTER AND POTTS MODELS

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ABSTRACT. We prove that for $q \geq 1$, there exists $r(q) < 1$ such that for $p > r(q)$, the number of points in large boxes which belongs to the infinite cluster has a normal central limit behaviour under the random cluster measure $\phi_{p,q}$ on \mathbb{Z}^d , $d \geq 2$. Particularly, we can take $r(q) = p_g^*$ for $d = 2$, which is commonly conjectured to be equal to p_c . These results are used to prove a q -dimensional central limit theorems relative to the fluctuation of the empirical measures for the ground Gibbs measures of the q -state Potts model at very low temperature and the Gibbs measures which reside in the convex hull of them. A similar central limit theorem is also given in the high temperature regime. Some particular properties of the Ising model are also discussed.

1. INTRODUCTION

The aim of this study is to answer to a natural question relative to Gibbs measures in the q -state Potts model: take a finite box in \mathbb{Z}^d and consider the frequencies of occurring of each of the q state in the box. It is obvious that the vector of empirical frequencies converges to a constant when the considered Gibbs measure is ergodic. Now, several natural questions arise. Two years ago, Cerf and Pisztora considered the difficult problem of large deviations [7]. We will consider here the problem of having a central limit theorem with a standard renormalization. To this aim, we will use the road of the Fortuin-Kasteleyn random cluster measure. In the last decade, consequent progresses have been made in the study of this model – see Grimmett [16] for a large panorama –, and it also appeared that in most cases, the shortest way to results relative to the ferromagnetic Potts model encounters random cluster model – see the review of Häggström [17] for a self-contained introduction to these relations.

Roughly speaking, we can say that a realization of the q -state Potts model with free boundary conditions in a finite box is a random coloring of the vertices of a realization of a free random-cluster measure in the box Λ , with the constraint that connected components are mono-color. Actually, we can consider the Potts model as the restriction to its vertices of a measure on “colored graphs”: there is randomness on the set of open bonds and also on the color of vertices, with the condition that connected components are mono-color. Biskup, Borgs, Chayes and Kotecký have shown that this approach could also be fruitful in the study of Potts models with external fields [4]. It leads them to introduce a generalization of the FK random cluster measure.

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It is not difficult to guess that the presence (or the absence) of an infinite cluster strongly modifies the fluctuation of the empiric repartitions. So it appears that the study of the supercritical case – which is, of course, the most interesting one – necessitates some knowledge about the normal fluctuations of the size of the intersection of the infinite cluster with large boxes. We made some progress in this aim but did not success to have such a result in the whole supercritical region. This gap in the spectrum of results, and the intuition we have that the random coloring scheme had already much to bring, lead us to adopt the following organization for our paper.

- In a first part, we prove that for each $q \geq 1$, there exists $r(q) < 1$ such that for $p > r(q)$, the number of points in large boxes which belongs to the infinite cluster has a normal central limit behaviour under the random cluster measure $\phi_{p,q}$. The result is much better on the square lattice: in this case, we can take $r(q) = p_g^*$, which is commonly conjectured to be equal to p_c – it is even proved for $q = 1, 2$ and $q \geq 22$.
- In a second part, we prove a d -dimensional central limit theorem for the fluctuation of the empiric repartitions of colors in a coloring model, that is model where the connected components of a random graph are paint independently. The random graph measure is supposed to be ergodic, to have appropriate moments for the size of finite clusters and, in case it has infinite cluster, to satisfy to a central limit theorem for the fluctuations of the size of the intersection of the infinite cluster with large boxes. This generalizes and extends the result obtained by the author [10] for the fluctuations of the magnetization in the case where the random graph measure is the classical Bernoulli measure.
- In a third part, we combined these results to obtain central limit theorems for the q -state Potts model which may be summarized as follows: When $\beta < \beta_g = -\frac{1}{2} \ln(1 - p_g)$, the vector of empiric repartitions satisfy to an ordinary central limit theorem under the unique Gibbs measure. If $\beta > \beta_r$, we have a central limit theorem with a random centering under each Gibbs measure which is in the convex hull of the ground Gibbs measures. The limit is normal for the ground Gibbs measures, but not necessarily for their mixtures. The result is also better for $d = 2$ because $\beta_r = \beta_c$ as soon as $r(q) = p_c$. In the case of the Ising model, the limit is always Gaussian even for a mixture of ground Gibbs measures, so we can say that for $d = 2$ and $\beta \neq \beta_c$, the fluctuation of the empirical magnetization around their (random) limit magnetization are asymptotically normal, whatever Gibbs measure we consider.

2. NOTATIONS AND PRELIMINARIES

Graph theoretical notations

For $x \in \mathbb{Z}^d$, let us denote $\|x\| = \sum_{i=1}^d |x_i|$ and now consider the graph $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, with

$$\mathbb{E}^d = \{\{x, y\}; x, y \in \mathbb{Z}^d \text{ and } \|x - y\| = 1\}.$$

For $x \in \mathbb{Z}^d$ and $r \in [0, +\infty)$, we note $B(x, r) = \{y \in \mathbb{Z}^d; \|x - y\| \leq r\}$. If $e = \{x, y\} \in \mathbb{E}^d$, then x and y are called *neighbours*.

In the following, the expression "subgraph of \mathbb{L}^d " will always be employed for each graph of the form $G = (\mathbb{Z}^d, E)$ where E is a subset of \mathbb{E}^d . We denote by $\mathcal{S}(\mathbb{L}^d)$ the set of all subgraphs of \mathbb{L}^d .

Set $\Omega = \{0, 1\}^{\mathbb{E}^d}$. An edge $e \in \mathbb{E}^d$ is said to be *open* in the configuration ω if $\omega(e) = 1$, and *closed* otherwise.

There is a natural bijection between $\mathcal{S}(\mathbb{L}^d)$ and Ω , that is $E \mapsto (1_{e \in E})_{e \in \mathbb{E}^d}$. Consequently, we sometimes identify $\mathcal{S}(\mathbb{L}^d)$ and Ω and say "random graph measure" rather than "measure on Ω ".

A *path* is a sequence $\gamma = (x_1, e_1, x_2, e_2, \dots, x_n, e_n, x_{n+1})$ such that x_i and x_{i+1} are neighbours and e_i is the edge between x_i and x_{i+1} . We will also sometimes describe γ only by the vertices it visits $\gamma = (x_1, x_2, \dots, x_n, x_{n+1})$ or by its edges $\gamma = (e_1, e_2, \dots, e_n)$. The number n of edges in γ is called the *length* of γ and is denoted by $|\gamma|$. Of particular interest are the *simple paths* for which the visited vertices are all distinct. We will also consider *cycles*, that are paths for which the visited vertices are all distinct, except that $x_1 = x_{n+1}$. A path is said to be *open* in the configuration ω if all its edges are open in ω .

The *clusters* of a configuration ω are the connected components of the graph induced on \mathbb{Z}^d by the open edges in ω . For x in \mathbb{Z}^d , we denote by $C(x)$ the cluster containing x . In other words, $C(x)$ is the set of points in \mathbb{Z}^d that are linked to x by an open path. We note $x \leftrightarrow y$ to signify that x and y belong to the same cluster. If A and B are subsets of \mathbb{Z}^d , $A \leftrightarrow B$ means that there exists $(x, y) \in A \times B$ with $x \leftrightarrow y$. We also note $x \leftrightarrow \infty$ to say that $|C(x)| = +\infty$. In the whole paper, we will note $I = \{x \in \mathbb{Z}^d; x \leftrightarrow \infty\}$.

We say that two bonds e and e' of \mathbb{E}^d are neighbours if $e \cap e'$ is not empty. It also gives a notion of connectedness in \mathbb{E}^d by the usual way.

For each subset Λ of \mathbb{Z}^d , we will note by $\partial\Lambda$ the boundary of Λ :

$$\partial\Lambda = \{y \in \Lambda^c; \exists x \in \Lambda \text{ with } \|x - y\| = 1\}$$

and \mathbb{E}_Λ the set of inner bonds of Λ :

$$\mathbb{E}_\Lambda = \{e \in \mathbb{E}^d; e \subset \Lambda\}.$$

Note that if Λ and Λ' are disjoint sets, then \mathbb{E}_Λ and $\mathbb{E}_{\Lambda'}$ are disjoint too.

For each $E \subset \mathbb{E}^d$, we denote by $\sigma(E)$ the σ -field generated by the projections $(\omega_e)_{e \in E}$.

When $\Lambda \subset \mathbb{Z}^d$, we also use the notation $\sigma(\Lambda)$ instead of $\sigma(\mathbb{E}_\Lambda)$.

We sometimes consider another set of bonds on \mathbb{Z}^d , that is

$$\mathbb{F}^d = \{\{x, y\}; x, y \in \mathbb{Z}^d \text{ and } \|x - y\|_\infty = 1\},$$

where $\|x\|_\infty = \max(|x_i|; 1 \leq i \leq d)$. If $e = \{x, y\} \in \mathbb{F}^d$, then x and y are called **-neighbours*. Similarly, we define the notion of **-paths*, **-cycles*, ** connected sets*,... exactly in the same way that for the graph \mathbb{L}^d .

For our central limit theorems, we will use boxes $(\Lambda_t)_{t \geq 1}$, with

$$\Lambda_t = \{x \in \mathbb{Z}^d; \|x\|_\infty \leq t\}.$$

Let X and S be arbitrary sets. Each $\omega \in X^S$ can be considered as a map from S to X . We will denote ω_Λ its restriction to Λ . Then, when A and B are two disjoint subsets of S and $(\omega, \eta) \in X^A \times X^B$, $\omega\eta$ denotes the concatenation of ω and η , that

is the element $z \in X^{A \cup B}$ such that

$$z_i = \begin{cases} \omega_i & \text{if } i \in A \\ \eta_i & \text{if } i \in B. \end{cases}$$

FKG inequalities

If ϕ is a probability measure and f, g two measurable functions, we note

$$\text{Cov}_\phi(f, g) = \int f g \, d\mu - \left(\int f g \, d\mu \right) \left(\int f g \, d\mu \right).$$

If A and B are measurable events, we also note $\text{Cov}_\phi(A, B) = \text{Cov}_\phi(\mathbb{1}_A, \mathbb{1}_B) = \phi(A \cap B) - \phi(A)\phi(B)$.

If X in an ordered set, we say that a measure ϕ on $X^S, \mathcal{B}(X^S)$ satisfy to the FKG inequalities if for each pair of increasing functions f and g , we have $\text{Cov}_\phi(f, g) \geq 0$.

An event A is said to be increasing (*resp.* decreasing) if $\mathbb{1}_A$ (*resp.* $1 - \mathbb{1}_A$) is an increasing function. Of course, if ϕ satisfy to the FKG inequalities and A and B are increasing events, we have $\text{Cov}_\phi(A, B) \geq 0$.

2.1. FK Random cluster measures.

Let $0 \leq p \leq 1$ and $q > 0$. For each configuration $\eta \in \Omega$ and each connected subset E of \mathbb{E}^d we define the random-cluster measure $\phi_{E,p,q}^\eta$ with boundary condition η on $(\Omega, \mathcal{B}(\Omega))$ by

$$\phi_{E,p,q}^\eta(\omega) = \begin{cases} \frac{1}{Z_{\Lambda,p,q}^\eta} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k_\Lambda(\omega_\Lambda \eta_{\Lambda^c})} & \text{if } \omega_{E^c} = \eta_{E^c}, \\ 0 & \text{otherwise,} \end{cases}$$

where $k(\omega)$ is the number of components of the graph in the configuration ω which intersect E . $Z_{E,p,q}^\eta$ is the renormalizing constant

$$Z_{E,p,q}^\eta = \sum_{\omega \in \{0,1\}^E} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega_E \eta_{E^c})}.$$

For each $b \in \{0,1\}$, we will simply denote by $\phi_{E,p,q}^b$ the measure $\phi_{E,p,q}^\eta$ corresponding to the configuration η which is such that $\eta_e = b$ for each $e \in \mathbb{E}^d$.

When Λ is a finite subset of \mathbb{Z}^d , we also use the notation $\phi_{\Lambda,p,q}$ instead of $\phi_{\mathbb{E}_\Lambda,p,q}$.

A probability measure ϕ on $(\Omega, \mathcal{B}(\Omega))$ is called a *random-cluster measure* with parameters p and q if for each measurable set A and each finite subset Λ of \mathbb{Z}^d , we have the D.L.R. condition:

$$\phi(A) = \int_\Omega \phi_{\Lambda,p,q}^\eta(A) \, d\phi(\eta)$$

or equivalently if for each finite subset Λ of \mathbb{Z}^d , we have the equation

$$\phi(\cdot | \sigma(\mathbb{E}^d \setminus \mathbb{E}_\Lambda))(\eta) = \phi_{\Lambda,p,q}^\eta \quad \phi - a.s.$$

The set of such measures is denoted by $\mathcal{R}_{p,q}$.

Let $b \in \{0,1\}$. If $(\Lambda_n)_{n \geq 1}$ is an increasing sequence of volumes tending to \mathbb{Z}^d , it is known that the sequence $\phi_{\Lambda_n,p,q}^b$ as a weak limit which does not depend of the sequence $(\Lambda_n)_{n \geq 1}$. We denote by $\phi_{p,q}^b$ this limit. The following facts are well known; refer to the recent summary of Grimmett [16] of complete references.

- $\phi_{p,q}^b$ is a translation invariant ergodic measure.
- $\phi_{p,q}^b \in \mathcal{R}_{p,q}$.
- If $q \geq 1$, $\phi_{p,q}^b$ satisfies to the FKG inequalities.

- Let us note $\theta^b(p, q) = \phi_{p,q}^b(0 \leftrightarrow \infty)$. There exists $p_c(q) \in (0, 1)$, such that for each $b \in \{0, 1\}$ we have $\theta^b(p, q) = 0$ for $p < p_c(q)$ and $\theta^b(p, q) > 0$ for $p > p_c(q)$.

Exponential bonds

Grimmett and Piza [13] also introduced another critical probability: Let us define

$$Y(p, q) = \limsup_{n \rightarrow \infty} \left\{ n^{d-1} \phi_{p,q}(0 \leftrightarrow \partial B(0, n)) \right\}$$

and $p_g(q) = \sup \{p : Y(p, q) < \infty\}$. We have $0 < p_g(q) \leq p_c(q)$, and it is believed that $p_g(q) = p_c(q)$ for all $q \geq 1$. They proved the following exponential bound:

Proposition 1. *Let $q \geq 1$, $d \geq 2$. For $p < p_g(q)$, there exists a constant $\gamma = \gamma(p, q) > 0$ with*

$$(1) \quad \phi_{p,q}(0 \leftrightarrow \partial B(0, n)) \leq e^{-\gamma n} \text{ for large } n.$$

Stochastic comparison

Let us first recall the concept of domination for finite measures on a partially ordered set E . We say that a probability measure μ dominates a probability measure ν , if

$$\int f d\nu \leq \int f d\mu$$

holds as soon as f is an increasing function. We also write $\nu \prec \mu$.

If $q' \geq q$, $q' \geq 1$ and $\frac{p'}{q'(1-p')} \geq \frac{p}{q(1-p)}$, then $\phi_{p,q}^0 \prec \phi_{p',q'}^0$.

Isolation cages

Let F be a connected subset of \mathbb{E}^d . We say that F is an isolation cage if $\mathbb{E}^d \setminus F$ as two connected components in \mathbb{L}^d . In this case, we denote by $\text{Int}(F)$ (*resp.* $\text{Ext}(F)$) the bounded (*resp.* unbounded) connected component of $\mathbb{E}^d \setminus F$ and $\overline{\text{Int}}(F) = \text{Int}(F) \cup F$ (*resp.* $\overline{\text{Ext}}(F) = \text{Ext}(F) \cup F$).

Let E be a finite connected subset of \mathbb{E}^d and F be an isolation cage.

Note for $r \in \{0, 1\}$, $C_r = \{\forall e \in F; \omega_e = r\}$.

Then, for each $\sigma(\overline{\text{Int}}(F) \cap E)$ measurable event M_1 and each $\sigma(\overline{\text{Ext}}(F) \cap E)$ measurable event M_2 , we have the decoupling property:

$$(2) \quad \forall (b, r) \in \{0, 1\}^2 \quad \phi_{p,q}^b(M_1 \cap M_2 | C_r) = \phi_{p,q}^b(M_1 | C_r) \phi_{p,q}^b(M_2 | C_r)$$

Moreover

$$\phi_{p,q}^b(M_1 | C_r) = \phi_{\text{Int}(F), p, q}^r(M_1).$$

2.2. The Potts model. Let us recall the definition of Gibbs measure in the context of the Potts model. Let $q \geq 2$. We note by E_q a set of cardinal q .

For a finite subset Λ of \mathbb{Z}^d , the Hamiltonian on the volume Λ is defined by

$$H_\Lambda = 2\beta \sum_{\substack{e=(x,y) \in \mathbb{E}^d \\ e \cap \Lambda \neq \emptyset}} \mathbb{I}_{\{\omega(x) \neq \omega(y)\}}.$$

Then, we can define for each bounded measurable function f and for each $\omega \in E_q^{\mathbb{Z}^d}$,

$$\Pi_\Lambda f(\omega) = \frac{1}{Z_\Lambda(\omega)} \sum_{\eta \in E_q^\Lambda} \exp(-H_\Lambda(\eta_\Lambda \omega_{\Lambda^c})) f(\eta_\Lambda \omega_{\Lambda^c}),$$

where

$$Z_\Lambda(\omega) = \sum_{\eta \in E_q^\Lambda} \exp(-H_\Lambda(\eta_\Lambda \omega_{\Lambda^c})).$$

For each ω , we will denote by $\Pi_\Lambda(\omega)$ the measure on $E_q^{\mathbb{Z}^d}$ which is associated to map $f \mapsto \Pi_\Lambda f(\omega)$. A measure μ on $E_q^{\mathbb{Z}^d}$ is said to be a Gibbs measure for the q -state Potts model at inverse temperature β when for each bounded measurable function f and each finite subset Λ of \mathbb{Z}^d , we have

$$E_\mu(f | (X_i)_{i \in \Lambda^c}) = \Pi_\Lambda f \quad \mu \text{ a.s.}$$

For each $r \in E_q$, let us denote by $\Pi_\Lambda(r)$ the measure $\Pi_\Lambda(\omega)$ where ω is the element of $E_q^{\mathbb{Z}^d}$ with $\omega_x = r$ for each $x \in \mathbb{Z}^d$. It is known that for each $\beta > 0$ and each $q \in \mathbb{Z}^d$, the sequence $(\Pi_\Lambda(r))_\Lambda$ converges when Λ tends to \mathbb{Z}^d . Let us denote by $\text{WPt}_{q,\beta,r}$ this limit. By the general theory of Gibbs measures, this limit is necessarily a Gibbs measure – see for example the reference book by Georgii [11]. Häggström, Jonasson and Lyons [19] gave a nice characterization of it:

Proposition 2. *Let $q \in \{2, 3, \dots\}$ and $p \in [0, 1]$. Pick a random edge configuration $X \in \{0, 1\}^{\mathbb{E}^d}$ according to the random-cluster measure $\phi_{p,q}^1$. Then, for each **finite** connected component \mathcal{C} of X independently, pick a spin uniformly from E_q , and assign this spin to all vertices of \mathcal{C} . Finally assign value r to all vertices of infinite connected components. The $E_q^{\mathbb{Z}^d}$ -valued random spin configuration arising from this procedure is then distributed according to the Gibbs measure $\text{WPt}_{q,\beta,r}$ for the q -state Potts model at inverse temperature $\beta := -\frac{1}{2} \log(1 - p)$.*

3. CENTRAL LIMIT THEOREM FOR THE RANDOM CLUSTER MEASURE

We begin with a general theorem which gives sufficient conditions for having a central limit theorem for the fluctuations of the size of the intersection of large boxes with the infinite clusters. This will tell us what sort of estimates about random cluster measures can help us.

Theorem 1. *Let ϕ be a translation-invariant ergodic measure on $\mathcal{S}(\mathbb{L}^d)$. We suppose that ϕ satisfy to the FKG inequalities and that we have $\theta_\phi = \phi(0 \leftrightarrow \infty) > 0$.*

For each $n \in \mathbb{Z}^d$ and $r > 0$, let us note the event $D_{n,r} = \{|C(n)| > r\}$. Suppose also that there exists a sequence $(r_n)_{n \in \mathbb{Z}^d}$ such that the following assumptions together hold:

•

$$(m) \quad \sum_{n \in \mathbb{Z}^d} P(+\infty > |C(0)| \geq r_n) < +\infty$$

•

$$(c) \quad \sum_{n \in \mathbb{Z}^d} \text{Cov}(D_{0,r_n}, D_{n,r_n}) < +\infty.$$

Then, we have

• (S^*)

$$\sigma_\phi^2 = \sum_{k \in \mathbb{Z}^d} (\phi_{p,q}(0 \leftrightarrow \infty \text{ and } k \leftrightarrow \infty) - \theta_\phi^2) < +\infty$$

- (CLT)

$$\frac{|\Lambda_n \cap I| - \theta_\phi |\Lambda_n|}{|\Lambda_n|^{1/2}} \implies \mathcal{N}(0, \sigma_\phi^2).$$

Proof.

$$|\Lambda_n \cap I(\omega)| - \theta_\phi |\Lambda_n| = \sum_{k \in \Lambda_n} f(T^k \omega),$$

where T^k is the translation operator defined by $T^k(\omega) = (\omega_{k+e})_{e \in \mathbb{E}^d}$ and $f = \mathbb{1}_{\{|C(0)| = +\infty\}} - \theta_\phi$. Moreover, f is an increasing function and ϕ satisfies the F.K.G. inequalities. Then, $(f(T^k \omega))_{k \in \mathbb{Z}^d}$ is a stationary random field of square integrable variables satisfying to the F.K.G. inequalities. Therefore, according to Newman [23], the Central Limit Theorem is true if we prove that the quantity

$$(3) \quad \sum_{k \in \mathbb{Z}^d} \text{Cov}(f, f \circ T^k)$$

is finite, which is just proving (S^*) .

Let us define $B = \{|C(0)| = +\infty\}$, and for each $n \in \mathbb{Z}^d$, $A_n = \{|C(n)| = +\infty\}$, $\tilde{A}_n = \{|C(n)| \geq r_n\}$ and $\tilde{B}_n = \{|C(0)| \geq r_n\}$. Since $B \subset \tilde{B}_n$ and $A_n \subset \tilde{A}_n$, one has $\mathbb{1}_{\tilde{B}_n} = \mathbb{1}_B + \mathbb{1}_{\tilde{B}_n \setminus B}$ and $\mathbb{1}_{\tilde{A}_n} = \mathbb{1}_{A_n} + \mathbb{1}_{\tilde{A}_n \setminus A_n}$. It follows that

$$\text{Cov}(\mathbb{1}_{\tilde{A}_n}, \mathbb{1}_{\tilde{B}_n}) - \text{Cov}(\mathbb{1}_{A_n}, \mathbb{1}_B) = \text{Cov}(\mathbb{1}_{\tilde{A}_n \setminus A_n}, \mathbb{1}_{\tilde{B}_n}) + \text{Cov}(\mathbb{1}_{\tilde{B}_n \setminus B}, \mathbb{1}_{\tilde{A}_n}),$$

and hence that

$$\begin{aligned} |\text{Cov}(\mathbb{1}_{\tilde{A}_n}, \mathbb{1}_{\tilde{B}_n}) - \text{Cov}(\mathbb{1}_{A_n}, \mathbb{1}_B)| &\leq P(\tilde{A}_n \setminus A_n) + P(\tilde{B}_n \setminus B) \\ &\leq 2P(+\infty > |C(0)| \geq r_n) \end{aligned}$$

It follows that

$$\sigma_\phi^2 \leq 2 \sum_{n \in \mathbb{Z}^d} P(+\infty > |C(0)| \geq r_n) + \sum_{n \in \mathbb{Z}^d} |\text{Cov}(D_{0,r_n}, D_{n,r_n})| < +\infty.$$

□

The idea of using Newman's theorem to prove Central Limit Theorems for the density of infinite clusters in percolation models satisfying to the F.K.G. inequalities is not new, because it has already been pointed out by Newman and Schulman [24, 25] that $(S^*) + FK G \implies (CLT)$. The interest of this theorem is that it splits a problem about infinite clusters into two problems relative to finite clusters:

- The existence of sufficiently high moments
- A control of the correlation for the appearance of reasonably large clusters in two points which are separated by a large distance – note that we can rewrite $\text{Cov}(D_{0,r_n}, D_{n,r_n})$ as $\text{Cov}(D_{0,r_n}^c, D_{n,r_n}^c)$.

Example: In the case of Bernoulli percolation, the central limit theorem holds for $p > p_c$.

Proof. Simply take $r_n = \|n\|/3$. The convergence (m) follows for example by the result of Kesten and Zhang [21]: there exists $\eta(p) > 0$ such that

$$\forall n \in \mathbb{Z}_+ \quad P(|C(0)| = n) \leq \exp(-\eta(p)n^{(d-1)/d}).$$

Of course, such a sharp estimate is not necessary for our purpose. Estimates derived from Chayes, Chayes and Newman [9], and from Chayes, Chayes, Grimmett, Kesten

and Schonmann [8] would have been sufficient. The convergence of (c) is an evidence since D_{0,r_n} and D_{n,r_n} are independent for $\|n\| > 12$. \square

3.1. The case of dimension two. Let $\mathbb{Z}_*^2 = \mathbb{Z}^2 + (1/2, 1/2)$. For each bond $e = \{a, b\}$ of \mathbb{L}^2 (*resp.* \mathbb{L}_*^2), let us denote by $s(e)$ the only subset $\{i, j\}$ of \mathbb{Z}_*^2 (*resp.* \mathbb{Z}^2) such that the quadrangle $aibj$ is a square. s is clearly an involution. Let us also define $\mathbb{L}_*^2 = (\mathbb{Z}_*^2, \mathbb{E}_*^2)$, where $\mathbb{E}_*^2 = \{s(e); e \in \mathbb{E}^2\}$. It is easy to see that \mathbb{L}_*^2 is isomorphic to \mathbb{L}^2 .

For finite $A \subset \mathbb{Z}_*^2$, we denote by $\mathcal{P}\text{eierls}(A)$ the Peierls contour associated to A , that is

$$\mathcal{P}\text{eierls}(A) = \{e \in \mathbb{E}^2; \mathbb{1}_A \text{ is not constant on } s(e)\}.$$

There exists a finite family of cycles and paths – the so-called Peierls contours – such that $\mathcal{P}\text{eierls}(A)$ is the set of vertices visited by them. It is known that, provided that A is a bounded connected subset of \mathbb{L}_*^2 , there exists a unique cycle $\Gamma(A)$ which is a Peierls contour and surrounds A .

Note that in the two-dimensional lattice, the set of bonds of a cycle forms an isolation cage. So if γ is a cycle, we will simply denote by $\overline{\text{Int}}(\gamma)$ the set $\overline{\text{Int}}(E)$, where E is the set of bonds of γ .

Consider now the map

$$\begin{aligned} \{0, 1\}^{\mathbb{E}^2} &\rightarrow \{0, 1\}^{\mathbb{E}_*^2} \\ \omega &\mapsto \omega^* = (1 - \omega_{s(e)})_{e \in \mathbb{E}_*^2} \end{aligned}$$

For $\eta \in \{0, 1\}^{\mathbb{E}_*^2}$, we also denote by η^* the only $\omega \in \{0, 1\}^{\mathbb{E}^2}$ such that $\omega^* = \eta$.

For each subset A of $\{0, 1\}^{\mathbb{E}_*^2}$ (*resp.* $\{0, 1\}^{\mathbb{E}^2}$), we denote by A^* the set $\{\omega^*; \omega \in A\}$.

The following planar duality between planar random cluster measures is now well known: let us define p^* to be the unique element of $[0, 1]$ which satisfies to $F(p)F(p^*) = 1$, with $F(x) = \frac{1}{\sqrt{q}} \frac{x}{1-x}$. and also define a map t by

$$\begin{aligned} \{0, 1\}^{\mathbb{E}^2} &\rightarrow \{0, 1\}^{\mathbb{E}_*^2} \\ \omega &\mapsto (\omega_{e+(1/2, 1/2)})_{e \in \mathbb{E}_*^2} \end{aligned}$$

Then, for each $p \in [0, 1]$, $b \in \{0, 1\}$ and each event A , we have

$$\phi_{p,q}^b(A) = t\phi_{p^*,q}^{1-b}(A^*).$$

Let us note $r(q) = p_g(q)^*$. Since $p_g(q) > 0$, we have $r(q) < 1$. Note that it is believed that $p_g(q) = p_c(q)$. As was noted by Grimmett and Piza [13], the fact that $p_g(q) = p_c(q)$ would imply that p_c is the solution of the equation $x = x^*$, *i.e.* $p_c = \frac{\sqrt{q}}{1+\sqrt{q}}$. So, it follows that we have $r(q) = p_c(q)$ provided that $p_g(q) = p_c(q)$. When $d = 2$, this widely believed conjecture has already be proved for $q = 1, 2$ and $q \geq 22$ – see the Saint-Flour notes by Grimmett [15].

Lemma 1. *Let $d = 2$ and $p < p_g(q)$. There exists $K \in (0, +\infty)$ with*

$$\forall n \in \mathbb{Z}_+ \quad \phi_{p,q}(|C(0)| \geq n) \leq K \exp(-\gamma(p, q)\sqrt{n/2}).$$

Proof. Suppose that $n \geq 16$ and denote by r the integer part of $\sqrt{n/2} - 1$. Let $T = \{k \in \mathbb{Z}_+; C(0) \cap \partial B(0, k) \neq \emptyset\}$ and $R = \max T$.

Suppose $|C(0)| \geq n$: we have $C(0) \subset B(0, R)$, so

$$n \leq |C(0)| \leq |B(0, R)| = 1 + 2R(R + 1).$$

It follows that $r \leq R$. Since $C(0)$ is connected, we have $0 \leftrightarrow \partial B(0, r)$. Then $\phi_{p,q}(|C(0)| \geq n) \leq \phi_{p,q}(0 \leftrightarrow \partial B(0, r))$. The result follows then from Proposition 1. \square

When $d = 2$, it is known that $p_g \leq p_c \leq \frac{\sqrt{q}}{1+\sqrt{q}}$. It follows that $p_g^* \geq p_c^* \geq \frac{\sqrt{q}}{1+\sqrt{q}}$. Then $(p > p_g^*) \implies (p \neq p_c)$. Since it is known that $\mathcal{R}_{p,q}$ is a single when $p \neq p_c$ and $d = 2$, it follows that there is an unique random cluster measure for $p > r(q)$. Then, we simply write $\phi_{p,q}$ without any superscript.

Lemma 2. *We suppose here that $d = 2$. For each $p > r(q)$, there exists $K \in (0, +\infty)$ with*

$$\forall n \geq 1 \quad \phi_{p,q}(|C(0)| = n) \leq K n e^{-\gamma(p^*, q) \sqrt{n}}.$$

Note that $\gamma(p^, q) > 0$.*

Proof. We use here a duality argument. Let $p > r(q)$ and note $A = \{|C(0)| = n\}$. We have $\phi_{p,q}(A) = t\phi_{p^*,q}(A^*)$. In this case

$$t^{-1}(A^*) = \left\{ \begin{array}{l} \text{there exists at least one open cycle surrounding } (0, 1/2) \\ \text{Those of these cycles which minimizes the distance to } (0, 1/2) \\ \text{surrounds exactly } n \text{ closed bonds.} \end{array} \right\}.$$

The number of bounds used by this cycle is at least $2n + 2$. Moreover, the position of the first intersection of this cycle with the positive x -axis is at most n . So

$$t^{-1}(A^*) \subset \bigcup_{k=1}^n \{|C(ke_1)| \geq 2n\}.$$

It follows then from lemma 1 that

$$\phi_p(A) = \phi_{p^*,q}(A^*) \leq K n e^{-\gamma(p^*, q) \sqrt{n}}.$$

\square

The goal of the next lemma is to bound the covariance of two decreasing events who are defined by the state of the bonds in two boxes separated by a large distance. It is clear that it does not pretend to originality and that its use could have been replaced by those of an analogous result of the literature, *e.g.* Theorem 3.4 of Alexander [3] joined to its Remark 3.5. Nevertheless, we preferred to present our lemma because its proof is rather short and allows an instructive comparison with the case of an higher dimension which will be studied after.

Lemma 3. *Let $q \geq 1$. For each $p > r(q)$, there exists $C > 0$ and $\alpha > 0$ such that for each couple of boxes Λ_1 and Λ_2 and each pair of monotone events A and B , with A (resp. B) $\sigma(\Lambda_1)$ (resp. $\sigma(\Lambda_2)$) measurable, we have*

$$|Cov_\phi(A, B)| = |\phi_{p,q}(A \cap B) - \phi_{p,q}(A)\phi_{p,q}(B)| \leq C |\partial \Lambda| e^{-\alpha d(\Lambda_1, \Lambda_2)}.$$

Proof. Since $Cov_\phi(A, B) = -Cov_\phi(A^c, B) = Cov_\phi(A^c, B^c) = -Cov_\phi(A, B^c)$, we can assume that A and B are decreasing events. We can also assume without loss of generality that $\Lambda_1 = \{-n, \dots, n\} \times \{-p, \dots, p\}$. Put $\Lambda_1^* = \{-n + 1/2, \dots, n - 1/2\} \times \{-p + 1/2, \dots, p - 1/2\}$. For $x \in \mathbb{Z}_*^2$ and $\omega \in \Omega$, let us define $C^*(x)$ to be

the connected component of x in the configuration ω^* . Let now be F the random set defined by

$$F = \Gamma\left(\bigcup_{y \in \Lambda_1^*} C^*(y)\right)$$

and note $V = \{\overline{\text{Int}}(F) \cap \Lambda_2 = \emptyset\}$. The following facts are elementary, but relevant:

- For every curve γ surrounding the origin, the event $\{F = \gamma\}$ is $\sigma(\overline{\text{Int}}(\gamma))$ -measurable.
- For any subset γ of \mathbb{Z}^d , $\{F = \gamma\} \subset W_\gamma = \{\forall e \in \gamma; \omega_e = 1\}$.

Remember that if T is a $\sigma(\overline{\text{Ext}}(\gamma))$ -measurable event and R a $\sigma(\overline{\text{Int}}(\gamma))$ -measurable event, we have the following decoupling property:

$$\phi_{p,q}(R \cap T | W_\gamma) = \phi_{p,q}(R | W_\gamma) \phi_{p,q}(T | W_\gamma)$$

So, if $\overline{\text{Int}}(\gamma) \cap \Lambda_2 = \emptyset$ we have

$$\begin{aligned} \phi_{p,q}(A \cap B \cap \{F = \gamma\}) &= \phi_{p,q}(A \cap B \cap \{F = \gamma\} \cap W_\gamma) \\ &= \phi_{p,q}(W_\gamma) \phi_{p,q}(A \cap \{F = \gamma\} \cap B | W_\gamma) \\ &= \phi_{p,q}(W_\gamma) \phi_{p,q}(A \cap \{F = \gamma\} | W_\gamma) \phi_{p,q}(B | W_\gamma) \\ &= \phi_{p,q}(A \cap \{F = \gamma\} \cap W_\gamma) \phi_{p,q}(B | W_\gamma) \\ &= \phi_{p,q}(A \cap \{F = \gamma\}) \phi_{p,q}(B | W_\gamma) \\ &\leq \phi_{p,q}(A \cap \{F = \gamma\}) \phi_{p,q}(B) \end{aligned}$$

If we sum over suitable values of γ , we get

$$\phi_{p,q}(A \cap B \cap V) \leq \phi_{p,q}(A \cap V) \phi_{p,q}(B) \leq \phi_{p,q}(A) \phi_{p,q}(B).$$

Since A and B are decreasing events, they are positively correlated, then

$$0 \leq \phi_{p,q}(A \cap B) - \phi_{p,q}(A) \phi_{p,q}(B) \leq \phi_{p,q}(A \cap B \cap V^c) \leq \phi_{p,q}(V^c).$$

Since

$$V^c \subset \bigcup_{y \in \partial \Lambda^*} \{y \leftrightarrow \partial B(y, d(\Lambda_1, \Lambda_2))\},$$

the result follows from the inequality of Grimmett and Piza. \square

3.2. The case of general d . The goal of the next lemma is also to bound the covariance of two positive events who are defined by the state of the bonds in two boxes separated by a large distance.

Unlike the proof of lemma 3, it can not use duality arguments. We nevertheless attempt to present this proof in a form which is as close as possible of those of lemma 3 to highlight the differences and the similarities between them.

Not that it is somewhat inspired by the proof of Grimmett [14] for the uniqueness of the random-cluster when p is large.

Lemma 4. *Let $q \geq 1$. For each $p > \frac{q}{q+4-d}$, there exists $\alpha(p, q) > 0$ such that for each couple of finite connected volumes Λ_1 and Λ_2 and each pair of monotone events A and B , with A (resp. B) $\sigma(\Lambda_1)$ (resp. $\sigma(\Lambda_2)$) measurable, we have*

$$|Cov_\phi(A, B)| = |\phi_{p,q}(A \cap B) - \phi_{p,q}(A) \phi_{p,q}(B)| \leq C |\partial \Lambda| e^{-\alpha d(\Lambda_1, \Lambda_2)}.$$

Proof. We begin by a topological remark: let A be a finite connected subset of \mathbb{Z}^d and consider the $*$ -connected components of ∂A : it is not difficult to see that there is exactly one of the connected components, say B , which surrounds A . Let us also define $W(A) = \{e \in \mathbb{E}^d; e \cap B \neq \emptyset\}$. Since B is $*$ -connected, $W(A)$ is connected in \mathbb{L}^d ; $W(A)$ also surrounds A . Note that $W(A)$ is the analogous to a surrounding Peierls contour in the two dimensional lattice.

Now suppose as previously that A and B are decreasing events. Given a configuration ω , say that a point $x \in \mathbb{Z}^d$ is wired if each of the 2^d bonds attached to x satisfy to $\omega_e = 1$. Otherwise, say that x is free. Let us define $D(\omega)$ to be the set of points in $\mathbb{Z}^d \setminus \Lambda_1$ which can be connected to Λ_1 using only free vertices – the origin of the path in Λ_1 does not need to be free. By definition of D , $(\Lambda_1 \cup D)$ is a connected set.

Let us note $F = W(D)$ and $V = \{\Lambda_2 \cap D = \emptyset\}$.

Note that V is an increasing event.

The following facts are elementary, but relevant:

- For every isolation cage γ surrounding Λ_1 , the event $\{F = \gamma\}$ is $\sigma(\overline{\text{Int}}(\gamma))$ -measurable.
- For any subset γ of \mathbb{E}^d , $\{F = \gamma\} \subset W_\gamma = \{\forall e \in \gamma; \omega_e = 1\}$.

Remember that if T is a $\sigma(\overline{\text{Ext}}(\gamma))$ -measurable event and R a $\sigma(\overline{\text{Int}}(\gamma))$ -measurable event, we have the following decoupling property:

$$\phi_{p,q}^0(R \cap T | W_\gamma) = \phi_{p,q}^0(R | W_\gamma) \phi_{p,q}^0(T | W_\gamma)$$

So, if γ does not touch nor surround Λ_2 , we have

$$\begin{aligned} \phi_{p,q}^0(A \cap B \cap \{F = \gamma\}) &= \phi_{p,q}^0(A \cap B \cap \{F = \gamma\} \cap W_\gamma) \\ &= \phi_{p,q}^0(W_\gamma) \phi_{p,q}^0(A \cap \{F = \gamma\} \cap B | W_\gamma) \\ &= \phi_{p,q}^0(W_\gamma) \phi_{p,q}^0(A \cap \{F = \gamma\}) \phi_{p,q}^0(B | W_\gamma) \\ &= \phi_{p,q}^0(A \cap \{F = \gamma\} \cap W_\gamma) \phi_{p,q}^0(B | W_\gamma) \\ &= \phi_{p,q}^0(A \cap \{F = \gamma\}) \phi_{p,q}^0(B | W_\gamma) \\ &\leq \phi_{p,q}^0(A \cap \{F = \gamma\}) \phi_{p,q}^0(B) \end{aligned}$$

If we sum over suitable values of γ , we get

$$\phi_{p,q}^0(A \cap B \cap V) \leq \phi_{p,q}^0(A \cap V) \phi_{p,q}^0(B) \leq \phi_{p,q}^0(A) \phi_{p,q}^0(B).$$

Since A and B are decreasing events, they are positively correlated, then

$$0 \leq \phi_{p,q}^0(A \cap B) - \phi_{p,q}^0(A) \phi_{p,q}^0(B) \leq \phi_{p,q}^0(A \cap B \cap V^c) \leq \phi_{p,q}^0(V^c).$$

Since V is an increasing event, we can use the stochastic domination of a product measure by $\phi_{p,q}^0$: $\phi_{r,1}^0 \prec \phi_{p,q}^0$, with $r = p/(p + (1-p)q)$, so $\phi_{p,q}^0(F^c) \leq \phi_{r,1}^0(F^c)$. Now, a Peierls-like counting argument gives

$$\phi_{r,1}^0(F^c) \leq |\partial \Lambda_1| (2^d - 1) (2^d (2^d - 1))^n (1 - r)^n,$$

where $n = d(\Lambda_1, \Lambda_2) - 1$, which completes the proof. \square

Theorem 2. *For each $q \geq 1$, there exists $r(q) < 1$ such that, for $p > r(q)$, $\mathcal{R}_{p,q}$ consists in an unique measure $\phi_{p,q}$. Moreover, if we note $\theta(p, q) = \phi_{p,q}(0 \in I)$, we have*

$$\frac{|\Lambda_n \cap I| - \theta(p, q)|\Lambda_n|}{|\Lambda_n|^{1/2}} \Longrightarrow \mathcal{N}(0, \sigma_p^2),$$

where I is the infinite cluster for FK percolation and

$$\sigma_{p,q}^2 = \sum_{k \in \mathbb{Z}^d} (\phi_{p,q}(0 \leftrightarrow \infty \text{ and } k \leftrightarrow \infty) - \theta(p, q)^2).$$

Note also that we can take $r(1) = p_c(\mathbb{Z}^d)$, and $r(q) = p_g^*$ for $d = 2$.

Proof. The uniqueness of the random cluster measure for p close to 1 as be proved by Grimmett [14]. So it only remains to prove that it holds for $p > r(q)$ in the cases where we have announced a convenient value for $r(q)$. When $q = 1$, the uniqueness is obvious and we have already remarked that there was uniqueness for $d = 2$ and $p > p_g^*$.

Let us now prove the Central Limit Theorem. We will apply Theorem 1 to the sequence $r_n = \|n\|/4$.

Let us show that

$$\sum_{n \in \mathbb{Z}^d} P(+\infty > |C(0)| \geq \frac{\|n\|}{4})$$

converges.

- For $d \geq 3$ and p sufficiently close to 1, this follows from the estimate of Pisztora [26]: for each $b \in \{0, 1\}$ and each p which is sufficiently close to 1, there exist a constant $a = a(p, q)$ with

$$\forall n \geq 0 \quad \phi_{p,q}^b(|C| = n) \leq \exp(-an^{(d-1)/d}).$$

- For $d = 2$ and $p > p_g(q)^*$, it follows from our lemma 2.

Now, it remains to prove that

$$(4) \quad \sum_{n \in \mathbb{Z}^d} \text{Cov}(\mathbb{1}_{\tilde{A}_n}, \mathbb{1}_{\tilde{B}_n}) < +\infty,$$

with $\tilde{A}_n = \{|C(0)| \geq r_n\}$ and $\tilde{B}_n = \{|C(n)| \geq r_n\}$.

Put $\Lambda_n = B(n, \|n\|/3)$ and $\Lambda'_n = B(0, \|n\|/3)$. It is clear that \tilde{A}_n (resp. \tilde{B}_n) is $\sigma(\Lambda_n)$ (resp. $\sigma(\Lambda'_n)$) measurable. It is obvious that \tilde{A}_n and \tilde{B}_n are increasing events. Then, we can apply lemma 4. Since $d(\Lambda_n, \Lambda'_n) \geq n/3$, it follows that

$$0 \leq \text{Cov}(\mathbb{1}_{\tilde{A}_n}, \mathbb{1}_{\tilde{B}_n}) \leq Kn^{d-1}e^{-\frac{\alpha(p,q)}{3}n},$$

which forms a convergent series as soon as $p > p(q)$. When $d = 2$, the result follows similarly from lemma 3. \square

4. COLORING OF RANDOM CLUSTERS

If G is a subgraph of \mathbb{L}^d , $r \in \mathbb{R}$ and if ν is a probability measure on \mathbb{R} , we will define the color-probability $P^{G,\nu,r}$ as follows: $P^{G,\nu,r}$ is the unique measure on $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d}))$ under which the canonical projections X_i – defined, as usual by $X_i(\omega) = \omega_i$ – satisfy

- For each $i \in \mathbb{Z}^d$, the law of X_i is
 - ν if $|C(i)| < +\infty$.
 - δ_r otherwise.

- For each independent set $S \subset \mathbb{Z}^d$, the variables $(X_i)_{i \in S}$ are independent.
- For each connected set $S \subset \mathbb{Z}^d$, the variables $(X_i)_{i \in S}$ are identical.

Let ϕ be a measure on $\mathcal{S}(\mathbb{L}^d)$.

The randomized color-measure associated to ϕ is defined by

$$P^{\phi, \nu, r} = \int_{\mathcal{S}(\mathbb{L}^d)} P^{G, \nu, r} d\phi(G).$$

Of course, our aim is to specialize ϕ to a Fortuin-Kastelein measure $\phi = \phi_{p,q}^b$, with $b \in \{0, 1\}$, but the reasoning that we will make only depend from the existence of

- stationarity and ergodicity of ϕ
- moments condition for the size of finite clusters
- existence of a central limit theorem for size of the intersection of infinite clusters with large boxes

In order to make easier the later use of these results according to the progress that would be made in random-cluster or related models, we try to expose our results in the most possible generality, and then to apply them according to the results that we have nowadays.

To motivate this work, let us give some examples of models covered by randomized color-measures when $\phi = \phi_{p,q}^b$.

- The case $q = 1$ is a generalization of the divide and color model of Häggström [18], which has already been studied in a earlier paper of the author [10].
- The most celebrated of the randomized color-measure is obtained when $q \geq 2$ is an integer and $\nu = \frac{1}{q}(\delta_1 + \delta_2 + \dots \delta_q)$. In this case $P^{\phi, \nu, r}$ is the Gibbs measure $\text{Wt}_{q, \beta, r}$ for the q -state Potts model on \mathbb{Z}^d at inverse temperature $\beta := -\frac{1}{2} \log(1 - p)$, according to Proposition 2. It includes of course the case of the Ising model.
- If n_1, n_2, \dots, n_s are positive integers with $n_1 + n_2 + \dots + n_s = q$ and we take $\nu = \frac{1}{q}(n_1 \delta_1 + n_2 \delta_2 + \dots + n_s \delta_s)$, we obtain a fuzzy Potts model. It obviously follows from the previous example and the definition of a fuzzy Potts model.

In all this section, we will suppose that ϕ satisfy to the following assumptions:

- (E): ϕ is a translation-invariant ergodic measure on $\mathcal{S}(\mathbb{L}^d)$.
- (M): $\exists \alpha > d \quad \sum_{k=1}^{+\infty} k^\alpha P(|C(0)| = k) < +\infty$.

When $\theta_\phi = \phi(0 \leftrightarrow \infty) > 0$, the following assumption will also be considered:

$$(CLT) : \exists \sigma_\phi^2 > 0, \frac{|\Lambda_n \cap I| - \theta_\phi |\Lambda_n|}{|\Lambda_n|^{1/2}} \Longrightarrow \mathcal{N}(0, \sigma_\phi^2).$$

We begin with a general property of randomized color-measures.

Theorem 3. $P^{\phi, \nu, z}$ is translation invariant and the action of \mathbb{Z}^d on $P^{\phi, \nu, z}$ is ergodic.

Proof. Let $\Omega = \{0, 1\}^{\mathbb{E}^d}$, $\Omega_t = [0, 1]^{\mathbb{Z}^d}$, $\Omega_c = \mathbb{R}^{\mathbb{Z}^d}$, $\Omega_3 = \Omega \times \Omega_t \times \Omega_c$ and consider the action of \mathbb{Z}^d on $(\Omega_3, \mathcal{B}(\Omega_3), \phi \otimes U[0, 1]^{\otimes \mathbb{Z}^d} \otimes \nu^{\otimes \mathbb{Z}^d})$. Since $\phi \otimes U[0, 1]^{\otimes \mathbb{Z}^d} \otimes \nu^{\otimes \mathbb{Z}^d}$ is a direct product of an ergodic measure by two mixing measures, it follows that the action of \mathbb{Z}^d on Ω_3 is ergodic – see Brown [5] for instance.

Let us define $f : \Omega_3 \rightarrow \mathbb{R}$ by $f(\omega, \omega_t, \omega_c) = z$ if $|C(0)(\omega)| = +\infty$ or if exists $x, y \in C(0)(\omega)$, with $x \neq y$ and $\omega_t(y) = \omega_t(x)$. Otherwise, we define $f(\omega, \omega_t, \omega_c)$ to

be equal to $\omega_c(x)$, where x is the unique element of $C(0)(\omega)$ such that $(\omega_t)(x) = \max\{\omega_t(y); y \in C(0)\}$. Now, if we define $(X_k)_{k \in \mathbb{Z}^d}$, by $X_k(\omega, \omega_t, \omega_c) = f(\theta_k(\omega, \omega_t, \omega_c))$, it is not difficult that the law of $(X_k)_{k \in \mathbb{Z}^d}$ under $\phi \otimes U[0, 1]^{\otimes \mathbb{Z}^d} \otimes \nu^{\otimes \mathbb{Z}^d}$ is $P^{\phi, \nu, r}$. Since a factor of an ergodic system is an ergodic system – see also Brown [5] –, it follows that $P^{\phi, \nu, z}$ is ergodic under the action of \mathbb{Z}^d . \square

4.1. Normal fluctuations of sums for color-measures.

Theorem 4. *Suppose that ϕ satisfy to (E) and (M) and let ν be a probability measure on \mathbb{R} with a second moment. We put $m = \int_{\mathbb{R}} x d\nu(x)$ and $\sigma^2 = \int_{\mathbb{R}} (x - m)^2 d\nu(x)$.*

For ϕ -almost all G , the following holds:

$$\frac{1}{|\Lambda_n|^{1/2}} \left(\sum_{x \in \Lambda_n \setminus I} (X(x) - m) \right) \Rightarrow \mathcal{N}(0, \chi^f(\phi) \sigma^2)$$

where $I(G) = \{x \in \mathbb{Z}^d; x \leftrightarrow \infty\}$.

The following lemma will be very useful.

Lemma 5. *For each subgraph G of \mathbb{L}^d , let us denote by $(A_i)_{i \in J}$ the partition of G into connected components.*

Suppose that ϕ satisfy to (E) and (M). Then, we have for ϕ -almost all G :

$$\lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \sum_{i \in J; |A_i| < +\infty} |A_i \cap \Lambda_n|^2 = \chi^f(\phi),$$

where

$$\chi^f(\phi) = \sum_{k=1}^{+\infty} k \phi(|C(0)| = k).$$

Proof. Let us define $C'(x)$ by

$$C'(x) = \begin{cases} C(x) & \text{if } |C(x)| < +\infty \\ \emptyset & \text{otherwise} \end{cases}$$

and $C'_n(x) = C'(x) \cap \Lambda_n$.

It is easy to see that

$$(5) \quad \sum_{i \in J; |A_i| < +\infty} |A_i \cap \Lambda_n|^2 = \sum_{x \in \Lambda_n} |C'_n(x)|.$$

We have $|C'_n(x)| \leq |C(x)|$, and the equality holds if and only if $C'(x) \subset \Lambda_n$.

We have

$$\sum_{k=1}^{+\infty} n^{d-1} \phi(+\infty | C(0)| \geq n^{d/\alpha}) \leq \int_0^{+\infty} x^{d-1} \phi(+\infty > |C(0)| \geq n^{d/\alpha}) = \frac{1}{d} \int_{|C(0)| < +\infty} |C(0)|^\alpha d\phi.$$

It follows from a standard Borel-Cantelli argument that for μ_p -almost all G , there exists a (random) N such that

$$\forall n \geq N \quad \max_{x \in \Lambda_n} |C'(x)| \leq n^{d/\alpha}.$$

It follows that for each $x \in \Lambda_{n-n^{d/\alpha}}$, $C'(x)$ is completely inside Λ_n , and therefore $C'(x) = C'_n(x)$. Then,

$$\begin{aligned} \sum_{x \in \Lambda_{n-n^{d/\alpha}}} |C'(x)| &\leq \sum_{x \in \Lambda_n} |C'_n(x)| \leq \sum_{x \in \Lambda_n} |C'(x)| \\ \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_{n-n^{d/\alpha}}} |C'(x)| &\leq \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} |C'_n(x)| \leq \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} |C'(x)| \end{aligned}$$

By the ergodic Theorem, we have ϕ -almost surely:

$$\lim_{n \rightarrow +\infty} \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} |C'(x)| = \int |C'(0)| d\phi = \chi^f(\phi).$$

Since $\lim_{n \rightarrow +\infty} \frac{|\Lambda_{n-n^{d/\alpha}}|}{|\Lambda_n|} = 1$, the result follows. \square

Proof.

$$\sum_{x \in \Lambda_n \setminus I} (X(x) - m) = \sum_{i=1}^{+\infty} |C'_n(a_i)| (X(a_i) - m)$$

Then

$$\frac{1}{|\Lambda_n|^{1/2}} \sum_{x \in \Lambda_n \setminus I} (X(x) - m) = \left(\frac{s_n^2}{|\Lambda_n|} \right)^{1/2} \frac{1}{s_n} \sum_{i=1}^{+\infty} |C'_n(a_i)| (X(a_i) - m),$$

with

$$s_n^2 = \sum_{i=1}^{+\infty} |C'_n(a_i)|^2.$$

By lemma 5, we have for ϕ -almost all G $\lim_{n \rightarrow +\infty} \frac{s_n^2}{|\Lambda_n|} = \chi^f(\phi)$.

Now, it remains to prove that

$$(6) \quad \frac{1}{s_n} \sum_{i=1}^{+\infty} |C'_n(a_i)| (X(a_i) - m) \Rightarrow \mathcal{N}(0, \sigma^2).$$

Therefore, we will prove that for ϕ -almost all G , the sequence $Y_{n,k} = |C'_n(a_i)| (X(a_i) - m)$ satisfies the Lindeberg condition. For each $\epsilon > 0$, we have

$$\begin{aligned} \sum_{k=1}^{+\infty} \frac{1}{s_n^2} \int_{|Y_{n,k}| \geq \epsilon s_n} Y_{n,k}^2 dP^{G,\nu} &= \sum_{k=1}^{+\infty} \frac{|C'_n(a_k)|^2}{s_n^2} \int_{|C'_n(a_k)| |x| \geq \epsilon s_n} (x - m)^2 d\nu(x) \\ &\leq \int_{|x| \geq \frac{\epsilon}{\eta_n}} (x - m)^2 d\nu(x), \end{aligned}$$

with $\eta_n = \frac{\sup_{k \geq 1} |C'_n(a_k)|}{s_n}$. Thus, the Lindeberg condition is fulfilled if $\lim \eta_n = 0$.

But we have already seen that $s_n \sim (\chi^f(\phi) |\Lambda_n|)^{1/2}$, whereas $\sup_{k \geq 1} |C'_n(a_k)| = O(n^{d/\alpha}) = o(n^{d/2})$. This concludes the proof. \square

Theorem 5. *Let ϕ be a measure on $\mathcal{S}(\mathbb{L}^d)$ that satisfy to (E) and (M). Let ν be a probability measure on \mathbb{R} with a second moment. We put $m = \int_{\mathbb{R}} x d\nu(x)$ and $\sigma^2 = \int_{\mathbb{R}} (x - m)^2 d\nu(x)$. Let also $z \in \mathbb{R}$.*

- If $\theta_\phi = 0$, then we have under $P^{\phi, \nu, z}$

$$\frac{1}{|\Lambda_n|^{1/2}} \left(\sum_{x \in \Lambda_n \setminus I} (X(x) - m) \right) \implies \mathcal{N}(0, \chi^f(\phi) \sigma^2)$$

- If $(\theta_\phi > 0)$ and (CLT) hold, then we have under $P^{\phi, \nu, z}$

$$\frac{1}{|\Lambda_n|^{1/2}} \left(\sum_{x \in \Lambda_n} X(x) - ((1 - \theta_\phi)m + \theta_\phi r) |\Lambda_n| \right) \implies \mathcal{N}(0, \chi^f(\phi) \sigma^2 + (z - m)^2 \sigma_\phi^2).$$

Proof. In this proof, it will be useful to consider G as a random variable. Let $\Omega = \mathcal{S}(\mathbb{L}^d) \times \mathbb{R}^{\mathbb{Z}^d}$ and define the probability \mathbb{P} on $\mathcal{B}(\Omega)$ as a skew-product: for measurable $A \times B \in \mathcal{B}(\mathcal{S}(\mathbb{L}^d)) \times \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d})$, we have $\mathbb{P}(A \times B) = \int_A P^{G, \nu, z}(B) d\phi(G)$. Then, the law of the marginals G and X are $\mathbb{P}_G = \phi$ and $\mathbb{P}_X = P^{\phi, \nu, z}$. As usual, the letter \mathbb{E} will be used to denote an expectation – or a conditional expectation – under \mathbb{P} .

Rearranging the terms of the sum, we easily obtain

$$\left(\sum_{x \in \Lambda_n} X(x) - ((1 - \theta_\phi)m + \theta_\phi z) |\Lambda_n| \right) = \sum_{x \in \Lambda_n \setminus I} (X(x) - m) + (z - m)(|I \cap \Lambda_n| - |\Lambda_n| \theta_\phi)$$

We will now put

$$Q_n = \frac{1}{|\Lambda_n|^{1/2}} \left(\sum_{x \in \Lambda_n} X(x) - ((1 - \theta_\phi)m + \theta_\phi z) |\Lambda_n| \right),$$

and define

$$\forall t \in \mathbb{R} \quad \phi_{n,z}(t) = \mathbb{E} \exp(i Q_n t).$$

Thereby, we have

$$\phi_{n,z}(t) = \mathbb{E} \exp\left(-\frac{it}{|\Lambda_n|^{1/2}} \sum_{x \in \Lambda_n \setminus I} (X(x) - m) + (z - m)(|I \cap \Lambda_n| - |\Lambda_n| \theta_\phi)\right)$$

Conditioning by $\sigma(G)$ and using the fact that I is $\sigma(G)$ -measurable, we get $\phi_{n,z}(t) = \mathbb{E} f_n(t, \cdot) g_n((z - m)t, \cdot)$, with

$$\begin{aligned} f_n(t, \omega) &= \mathbb{E} \exp\left(-\frac{it}{|\Lambda_n|^{1/2}} \sum_{x \in \Lambda_n \setminus I} (X(x) - m) \middle| \sigma(G)\right) \\ &= \int \exp\left(-\frac{it}{|\Lambda_n|^{1/2}} \sum_{x \in \Lambda_n \setminus I(\omega)} (X(x) - m) \right) dP^{G(\omega), \nu} \end{aligned}$$

and

$$g_n(t, \omega) = \exp\left(-\frac{it}{|\Lambda_n|^{1/2}} (|I(\omega) \cap \Lambda_n| - |\Lambda_n| \theta_\phi)\right)$$

By Theorem 4 we have for each $t \in \mathbb{R}$ and $P^{\phi, \nu, z}$ -almost all ω : $\lim_{n \rightarrow +\infty} f_n(t, \omega) = \exp(-\frac{t^2}{2} \chi^f(\phi) \sigma^2)$. Then, by dominated convergence

$$\lim_{n \rightarrow +\infty} \mathbb{E} (f_n(t, \cdot) - \exp(-\frac{t^2}{2} \chi^f(\phi) \sigma^2)) g_n((z - m)t, \cdot) = 0.$$

Next

$$\begin{aligned} \lim_{n \rightarrow +\infty} \mathbb{E} f_n(t, \cdot) g_n((z-m)t, \cdot) &= \lim_{n \rightarrow +\infty} \exp\left(-\frac{t^2}{2} \chi^f(\phi) \sigma^2\right) \mathbb{E} g_n((z-m)t, \cdot) \\ &= \exp\left(-\frac{t^2}{2} \chi^f(\phi) \sigma^2\right) \exp\left(-\frac{t^2}{2} (z-m)^2 \sigma_p^2\right) \end{aligned}$$

where the last equality follows from Proposition 2. We have just proved that

$$\lim_{n \rightarrow \infty} \phi_{n,z}(t) = \exp\left(-\frac{t^2}{2} (\chi^f(\phi) \sigma^2 + (z-m)^2 \sigma_p^2)\right).$$

The result follows from the Theorem of Levy. \square

4.2. Fluctuation of the empirical vector associated to coloring models.

Definition Let q be an integer with $q \geq 2$. For every $r \in \{1, \dots, q\}$ and each vector $\nu \in \mathbb{R}_+^d$ with $\nu_1 + \dots + \nu_d = 1$, we denote by $\text{Color}_{\phi, \nu}^r$ the measure $P^{\phi, \nu, r}$.

For $\omega \in \{1, \dots, q\}^{\mathbb{Z}^d}$, and $\Lambda \subset \mathbb{Z}^d$, we note $n(\Lambda)(\omega) = (n_1(\Lambda)(\omega), \dots, n_q(\Lambda)(\omega))$, with $n_k(\Lambda)(\omega) = |\{x \in \Lambda; \omega_x = k\}|$.

Theorem 6. Let ϕ be a measure on $\mathcal{S}(\mathbb{L}^d)$ that satisfy to $(E), (M)$ and $\theta_\phi = 0$ or (CLT) . Let q be an integer with $q \geq 2$, $r \in \{1, \dots, q\}$, and $\nu \in \mathbb{R}_+^d$ with $\nu_1 + \dots + \nu_d = 1$. Then, under $\text{Color}_{\phi, \nu}^r$, we have

$$\frac{n_{\Lambda_t} - |\Lambda_t|((1 - \theta_\phi)\nu + \theta_\phi e_r)}{\sqrt{|\Lambda_t|}} \Rightarrow \mathcal{N}(0, C),$$

where C is the matrix associated to the quadratic form

$$Q(b) = \chi^f(\phi)(\langle D_\nu b, b \rangle - \langle \nu, b \rangle^2) + \sigma_\phi^2 \langle e_r - \nu, b \rangle^2,$$

with $D_\nu = \text{diag}(\nu_1, \dots, \nu_d)$. In other words, C is the matrix of the map

$$b \mapsto \chi^f(\phi)(D_\nu b - \langle \nu, b \rangle \nu) + \sigma_\phi^2 \langle e_r - \nu, b \rangle (e_r - \nu).$$

Proof. Let $b \in \mathbb{R}^d$ and note $Q_t = \frac{n_{\Lambda_t} - |\Lambda_t|((1 - \theta_\phi)\nu + \theta_\phi e_r)}{\sqrt{|\Lambda_t|}}$. For $x \in \mathbb{Z}^d$, let us note $Y_x = b_{X_x}$. We have

$$\begin{aligned} \langle n(\Lambda_t), q \rangle &= \sum_{k=1}^q n_k(\Lambda_t) b_k = \sum_{k=1}^q \sum_{x \in \Lambda_t} \delta_{X_x}(k) b_k \\ &= \sum_{x \in \Lambda_t} \sum_{k=1}^q \delta_{X_x}(k) b_k = \sum_{x \in \Lambda_t} b_{X_x} \\ &= \sum_{x \in \Lambda_t} Y_x \end{aligned}$$

Now put $m = \langle \nu, b \rangle$ and $z = b_r = \langle e_r, b \rangle$. We have

$$\langle Q_n, b \rangle = \frac{\left(\sum_{x \in \Lambda_t} Y_x \right) - |\Lambda_t|((1 - \theta_\phi)m + \theta_\phi z)}{\sqrt{|\Lambda_t|}}.$$

Now if we define μ to be the image of ν by $k \mapsto b_k$, it is not difficult to see that the mean of μ is m and that the law of $(Y_k)_{k \in \mathbb{Z}^d}$ under $\text{Color}_{\phi, \nu}^r$ is $P^{\phi, \mu, z}$. Then, it

follows from Theorem 5 that $\langle Q_n, b \rangle \implies \mathcal{N}(0, Q(b))$, with $Q(b) = \chi^f(\phi)\sigma^2 + (z - m)^2\sigma_\phi^2$, where σ^2 is the variance of ν . Finally, we get the explicit form

$$Q(b) = \chi^f(\phi)(\langle D_\nu b, b \rangle - \langle \nu, b \rangle^2) + \sigma_\phi^2 \langle e_r - \nu, b \rangle^2,$$

with $D_\nu = \text{diag}(\nu_1, \dots, \nu_d)$. Let L be a random vector following $\mathcal{N}(0, C)$, where C is the covariance matrix associated to Q . We have proved that

$$\forall b \in \mathbb{R}^d \langle Q_t, b \rangle \implies \langle L, b \rangle.$$

Using the theorem of Levy, it is easy to see that it is equivalent to say that $Q_t \implies L$. \square

We are now interested in having, for $\theta_\phi > 0$, a version of theorem 6 in which the observed quantity does not depend on r . There are several reasons to motivate such a theorem: if we want to use this central limit theorem to test if a concrete physical system conforms to this model (have in mind an Ising or a Potts model for instance), we have a priori no reason to guess the r phase of the underlying theoretical system. There is also a theoretical justification to wish for such a theorem: if we get a theorem which “does not depend” on r , it will be easy to transfer it to any measure which resides in the convex hull of the $(\text{Color}_{\phi, \nu}^r)_{1 \leq r \leq q}$.

Theorem 7. *Let ϕ be a measure on $\mathcal{S}(\mathbb{L}^d)$ that satisfy to (E), (M), $\theta_\phi > 0$ and (CLT). Let q be an integer with $q \geq 2$, $r \in \{1, \dots, q\}$, and $\nu \in \mathbb{R}_+^d$ with $\nu_1 + \dots + \nu_d = 1$. For $\Lambda \subset \mathbb{Z}^d$, we denote by R_Λ the element of $\{1, \dots, q\}$ which realizes the maximum of $(n_\Lambda(k) - |\Lambda|(1 - \theta_\phi)\nu(k))_{1 \leq k \leq q}$. Then, under $\text{Color}_{\phi, \nu}^r$, we have*

$$\frac{n_{\Lambda_t} - |\Lambda_t|((1 - \theta_\phi)\nu + \theta_\phi e_{R_{\Lambda_t}})}{\sqrt{|\Lambda_t|}} \implies \mathcal{N}(0, C),$$

where C is the matrix associated to the quadratic form

$$Q(b) = \chi^f(\phi)(\langle D_\nu b, b \rangle - \langle \nu, b \rangle^2) + \sigma_\phi^2 \langle e_r - \nu, b \rangle^2,$$

with $D_\nu = \text{diag}(\nu_1, \dots, \nu_d)$. In other words, C is the matrix of the map

$$b \mapsto \chi^f(\phi)(D_\nu b - \langle \nu, b \rangle \nu) + \sigma_\phi^2 \langle e_r - \nu, b \rangle (e_r - \nu).$$

Proof. Since $\text{Color}_{\phi, \nu}^r$ is ergodic $\frac{n_{\Lambda_t}}{|\Lambda_t|} = \frac{1}{|\Lambda_t|} \sum_{x \in \Lambda_t} e_{\omega_x}$ almost surely converges to the mean value of e_{ω_0} , that is $(1 - \theta_\phi)\nu + \theta_\phi e_r$. Then, we have the equivalent $n_{\Lambda_t} - |\Lambda_t|(1 - \theta_\phi)\nu \sim |\Lambda_t|\theta_\phi e_r$. It follows that $R_{\Lambda_t} = r$ if t is large enough. Now let g be a bounded continuous function on \mathbb{R}^d :

$$\begin{aligned} & \mathbb{E} g\left(\frac{n_{\Lambda_t} - |\Lambda_t|((1 - \theta_\phi)\nu + \theta_\phi e_{R_{\Lambda_t}})}{\sqrt{|\Lambda_t|}}\right) \\ = & \mathbb{E} g\left(\frac{n_{\Lambda_t} - |\Lambda_t|((1 - \theta_\phi)\nu + \theta_\phi e_r)}{\sqrt{|\Lambda_t|}}\right) \\ & + \mathbb{E} \left(g\left(\frac{n_{\Lambda_t} - |\Lambda_t|((1 - \theta_\phi)\nu + \theta_\phi e_{R_{\Lambda_t}})}{\sqrt{|\Lambda_t|}}\right) - g\left(\frac{n_{\Lambda_t} - |\Lambda_t|((1 - \theta_\phi)\nu + \theta_\phi e_r)}{\sqrt{|\Lambda_t|}}\right)\right). \end{aligned}$$

The first term of the sum converges to the integral of g under $\mathcal{N}(0, C)$ by Theorem 6 and the second one converges to 0 by dominated convergence. It follows that $\mathbb{E} g\left(\frac{n_{\Lambda_t} - |\Lambda_t|((1 - \theta_\phi)\nu + \theta_\phi e_{R_{\Lambda_t}})}{\sqrt{|\Lambda_t|}}\right)$ converges to the integral of g under $\mathcal{N}(0, C)$ for any

bounded continuous function g , which is exactly the weak convergence to $\mathcal{N}(0, C)$. \square

Theorem 8. *Let ϕ be a measure on $\mathcal{S}(\mathbb{L}^d)$ that satisfy to $(E), (M), \theta_\phi > 0$ and (CLT) . Let q be an integer with $q \geq 2$, $r \in \{1, \dots, q\}$, and $\nu \in \mathbb{R}_+^d$ with $\nu_1 + \dots + \nu_d = 1$. For $\Lambda \subset \mathbb{Z}^d$, we denote by R_Λ the element of $\{1, \dots, q\}$ which realizes the maximum of $(n_\Lambda(k) - |\Lambda|(1 - \theta_\phi)\nu(k))_{1 \leq k \leq q}$.*

Let γ be a measure on $\{1, \dots, q\}$ and $\Phi_\gamma = \int \text{Color}_{\phi, \nu}^r d\gamma(r)$. Then, under Φ_γ , we have

$$\frac{n_{\Lambda_t} - |\Lambda_t|((1 - \theta_\phi)\nu + \theta_\phi e_{R_{\Lambda_t}})}{\sqrt{|\Lambda_t|}} \Rightarrow \mu,$$

where μ is the law of $X + S(e_Z - \nu)$, where X, S and Z are independent, with $X \sim \mathcal{N}(0, C')$, $S \sim \mathcal{N}(0, \sigma_\phi^2)$ and $Z \sim \gamma$. C' is the matrix associated to the quadratic form

$$Q(b) = \chi^f(\phi)(\langle D_\nu b, b \rangle - \langle \nu, b \rangle^2)$$

with $D_\nu = \text{diag}(\nu_1, \dots, \nu_d)$. In other words, C' is the matrix of the map

$$b \mapsto \chi^f(\phi)(D_\nu b - \langle \nu, b \rangle \nu).$$

Proof. It just follows from Levy's theorem and a straightforward computation of characteristic function. \square

5. APPLICATIONS TO POTTS AND ISING MODELS

In the following, we will always take $\phi = \phi_{p,q}^b$, with $b \in \{0, 1\}$. We associate to p the inverse temperature $\beta = -\frac{1}{2} \ln(1 - p)$. Let us see when the assumptions of theorem 6 are satisfied.

- It is a well-known result that the assumption of ergodicity (E) is always satisfied.
- By the inequality of Grimmett and Piza, (M) is always satisfied when $p < p_g$, or equivalently $\beta < \beta_g = -\frac{1}{2} \ln(1 - p_g)$. (high temperature regime)
- By the inequality of Pisztor and our lemma 2, (M) holds when p is close enough to 1, or, equivalently, when β is large enough. (very low temperature regime)
- By our theorem 2, (CLT) holds when p is close enough to 1 ($p > r(q)$), or, equivalently, when β is large enough $\beta > \beta_r = -\frac{1}{2} \ln(1 - r(q))$. (very low temperature regime)

Remember that when $d = 2$, we have proved that (M) and (CLT) hold as soon as $p > r(q)$, which is known to be equal to $p_c(q)$ as soon as $p_g = p_c$. So, if the conjecture $p_g = p_c$ were proved, we would have the central limit theorem for each value of β , except for $\beta_c = -\frac{1}{2} \ln(1 - p_c) = \frac{1}{2} \ln(1 + \sqrt{q})$.

Empirical distributions of Potts models

We have already noticed that if we take $\phi = \phi_{p,q}^1$, with $q \geq 2$ and $\nu = \frac{1}{q}(\delta_1 + \delta_2 + \dots \delta_q)$, $\text{Color}_{\phi, \nu}^r$ is the Gibbs measure $\text{WPt}_{q, \beta, r}$ for the q -state Potts model on \mathbb{Z}^d at inverse temperature $\beta := -\frac{1}{2} \log(1 - p)$, according to Proposition 2.4 of Häggström, Jonasson and Lyons [19]. In the high temperature regime ($p < p_c(q)$), we have $\phi_{p,q}^1(0 \leftrightarrow \infty) = 0$, so $\text{WPt}_{q, \beta, r}$ does not depends on r . In fact, it is known that there is uniqueness of the Gibbs measure when $\phi_{p,q}^1(0 \leftrightarrow \infty) = 0$, so $\text{WPt}_{q, \beta, 1}$

is the unique Gibbs measure at inverse temperature β . We can now formulate a theorem corresponding to the high temperature regime:

Theorem 9. *Let $\beta < \beta_g$. There is a unique Gibbs measure for the q -state Potts model at inverse temperature β . If we note $p = 1 - \exp(-2\beta)$, we have the following results for the empirical distributions:*

$$\frac{n_{\Lambda_t} - |\Lambda_t|\nu}{\sqrt{|\Lambda_t|}} \Rightarrow \mathcal{N}(0, \frac{\chi(p, q)}{q^2}(qI - J)),$$

where J is the $q \times q$ matrix whose each entry is equal to 1, and

$$\chi(p, q) = \sum_{k=1}^{+\infty} k \phi_{p, q}^1(|C(0)| = k).$$

If $p > p_c$, then the Gibbs measures $\phi = \text{WPt}_{q, \beta, r}$ are all distinct (it can be seen as a consequence of the fact that $R_{\Lambda_t} \text{WPt}_{q, \beta, r}$ -almost surely converges to r). Since they are ergodic by theorem 3, they are affinely independent. Then, in the case $\beta > \beta_r$, we have obtained a central limit theorem for the empirical distribution for a q -dimensional convex set of Gibbs measures: the limit is Gaussian when $\phi = \text{WPt}_{q, \beta, r}$, in general it is not Gaussian for a convex combination of them.

Theorem 10. *Let $\beta > \beta_r$ and let Φ_γ be a Gibbs measure for the q states Potts model at inverse temperature β which can be written in the form $\Phi_\gamma = \int \text{WPt}_{q, \beta, r} d\gamma(r)$. For $\Lambda \subset \mathbb{Z}^d$, we denote by R_Λ the element of $\{1, \dots, q\}$ which realizes the maximum of $(n_\Lambda(k))_{1 \leq k \leq q}$. Let us note $p = 1 - \exp(-2\beta)$. Then, under Φ_γ , we have*

$$\frac{n_{\Lambda_t} - |\Lambda_t|((1 - \theta_\phi)\nu + \theta_\phi e_{R_{\Lambda_t}})}{\sqrt{|\Lambda_t|}} \Rightarrow \mu,$$

where μ is the law of $X + S(e_Z - \nu)$, when X, S and Z are independent, with $X \sim \mathcal{N}(0, \frac{\chi^f(p, q)}{q^2}(qI - J))$, $S \sim \mathcal{N}(0, \sigma_\phi^2)$ and $Z \sim \gamma$. J is the $q \times q$ matrix whose each entry is equal to 1, and

$$\chi^f(p, q) = \sum_{k=1}^{+\infty} k \phi_{p, q}^1(|C(0)| = k).$$

Remark: An interesting case of a convex combination is obtain when $\gamma = \nu$, with the notations of Theorem 8. Since we have uniqueness of the infinite cluster in the random cluster cluster, we can consider that the law of Φ_ν is obtained by coloring independently each connected component of the random cluster. Then, Φ_ν is just $\text{FPt}_{q, \beta}^{\mathbb{Z}^d}$ in the terminology of Proposition 2.3 of Häggström, Jonasson and Lyons [19].

Fluctuations of the magnetization in Ising models

In spite of the fact that μ is in general not Gaussian, we can observe an intriguing fact when $q = 2$, i.e. for the Ising model. In this case $S(e_Z - \nu) = \epsilon S \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$, with $\epsilon = (-1)^{\{z=1\}}$. But ϵS has the same law than S . It follows that μ does not depend on γ and is always Gaussian.

Note also that it is known that we have an exponential decay of the covariance in the Ising model at high temperature – the exact Ornstein-Zernike directional speed of decay has even be proved by Campanino, Ioffe and Velenik [6]. It follows

that we have $p_c = p_g$ or equivalently $\beta_c = \beta_g$. Since the value of the critical point when $d = 2$ is the fixed point of $x \mapsto x^*$, we have even $\beta_r = \beta_c$ when $d = 2$.

In this model, it is most relevant to formulate the result in term of the magnetization $m_\Lambda = n_\Lambda \cdot (1 - 1)$ rather than in terms of n_Λ .

Theorem 11. *Let $\beta > \beta_r$ and let Φ_γ be a Gibbs measure for the Ising model on $\{-1, +1\}^{\mathbb{Z}^d}$ at inverse temperature β which can be written in the form $\Phi_\gamma = \gamma \text{WPt}_{2,\beta,1} + (1-\gamma) \text{WPt}_{2,\beta,-1}$. Let us note $p = 1 - \exp(-2\beta)$ and $m_\Lambda = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \omega_x$. Then, under Φ_γ , we have*

$$\sqrt{|\Lambda_t|}(m_{\Lambda_t} - \text{sign}(m_\Lambda)\theta(p, 2)) \implies \mathcal{N}(0, \chi^f(p, 2) + \sigma_{p,2}^2),$$

where

$$\chi^f(p, q) = \sum_{k=1}^{+\infty} k \phi_{p,q}^1(|C(0)| = k).$$

Note that $\beta_r = \beta_c$ when $d = 2$.

Note that for $d = 2$, Theorem 11 covers the whole set of Gibbs measure at temperature $\beta > \beta_c$. Indeed, $\text{WPt}_{q,\beta,1}$ and $\text{WPt}_{q,\beta,-1}$ are known to be the only two extremal Gibbs measures when $d = 2$. (This celebrated result is due to Higuchi [20] and Aizenmann [2]. See also Georgii and Higuchi [12] for a modern proof.) It follows that every Gibbs measure is a convex combination of $\text{WPt}_{q,\beta,1}$ and $\text{WPt}_{q,\beta,-1}$. We also note that $\theta(p, 2)$ appears as the spontaneous magnetization in the “+” phase of the Ising model. Since the explicit expression of the spontaneous magnetization is known when $d = 2$ – see Abraham and Martin-Löf [1], Aizenman [2], and also the bibliographical notes by Georgii [11] for the whole long story of this result –, we get for $d = 2$ and $p \geq p_c$ the formula $\theta(p, 2) = (1 - (\sinh 2\beta)^{-4})^{1/8} = (1 - 16 \frac{(1-p)^4}{p^4(2-p)^4})^{1/8}$.

Of course we have a similar theorem in the high temperature regime $\beta < \beta_g = \beta_c$.

Theorem 12. *Let $\beta < \beta_c$ and let Φ be the unique Gibbs measure for the Ising model on $\{-1, +1\}^{\mathbb{Z}^d}$ at inverse temperature β . We note $p = 1 - \exp(-2\beta)$ and $m_\Lambda = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \omega_x$.*

Then, under Φ , we have

$$\sqrt{|\Lambda_t|}m_{\Lambda_t} \implies \mathcal{N}(0, \chi^f(p, 2)),$$

where

$$\chi^f(p, q) = \sum_{k=1}^{+\infty} k \phi_{p,q}^1(|C(0)| = k).$$

Note that for the Gibbs measures $\text{WPt}_{2,\beta,1}$ or $\text{WPt}_{2,\beta,-1}$, the central limit theorems could be proved without the machinery of the above section: since the Ising model satisfy to the F.K.G. inequalities, it follows from the theorem of Newman that is sufficient to prove that

$$\sum_{k \in \mathbb{Z}^d} \text{Cov}(\sigma_0, \sigma_k) < +\infty.$$

But it is not difficult to see that under $\text{WPt}_{2,\beta,1}$ or $\text{WPt}_{2,\beta,-1}$, we have $\text{Cov}(\sigma_0, \sigma_k) = \phi_{p,2}(0 \leftrightarrow k) - \phi_{p,2}(0 \leftrightarrow \infty)^2$. But $\phi_{p,2}(0 \leftrightarrow k) = \phi_{p,2}(0 \leftrightarrow k)$ by a finite cluster) +

$\phi_{p,2}(0 \leftrightarrow \infty, k \leftrightarrow \infty)$. Then

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} \text{Cov}(\sigma_0, \sigma_k) &= \sum_{k \in \mathbb{Z}^d} \phi_{p,2}(0 \leftrightarrow k \text{ by a finite cluster}) + \sum_{k \in \mathbb{Z}^d} \phi_{p,2}(0 \leftrightarrow \infty, k \leftrightarrow \infty) - \phi_{p,2}(0 \leftrightarrow \infty)^2 \\ &= \chi^f(p, 2) + \sigma_{p,2}^2 \end{aligned}$$

Nevertheless, when $\beta > \beta_r$, the Gibbs measure “with free boundary conditions” $\text{FPT}_{2,\beta}^{\mathbb{Z}^d}$ – which satisfy to the assumptions of theorem 11 – does not have finite susceptibility: in this case

$$\text{Cov}(\sigma_0, \sigma_k) = \mathbb{E} \sigma_0 \sigma_k = \phi_{p,2}(0 \leftrightarrow k) \geq \phi_{p,2}(0 \leftrightarrow \infty, k \leftrightarrow \infty) \geq \phi_{p,2}(0 \leftrightarrow \infty)^2 > 0,$$

so the series $\sum_{k \in \mathbb{Z}^d} \text{Cov}(\sigma_0, \sigma_k)$ diverges.

These results can be compared with a result of Martin-Löf [22]: he also proved a central limit theorem for the magnetization in Ising Models at very low temperature. Particularly, he relays the variance of the limiting normal measure to the second derivative at 0 of the thermodynamical function F . Nevertheless, his result is slightly different from ours, since he considers Gibbs measures in large boxes with boundary condition “+”, whereas we consider here infinite Gibbs measures under the “+” phase.

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